Variational representation of the nuclear norm:

Thm: $|||X|||_* = \min_{X=AB^T} |||A|||_F \cdot |||B|||_F = \min_{X=AB^T} \frac{1}{2}\left(|||A|||_F^2 + |||B|||_F^2\right)$

Proof: - Let $X = U\Sigma V^T$ be its SVD. Define $C = U\Sigma^{\frac{1}{2}}$ and $D = V\Sigma^{\frac{1}{2}}$, then $X = CD^T$. Moreover,

$$|||C|||_F^2 = \mathrm{trace}(C^TC) = \mathrm{trace}(\Sigma^{\frac{1}{2}}U^TU\Sigma^{\frac{1}{2}}) = \mathrm{trace}(\Sigma).$$

Similarly, $|||D|||_F^2 = \mathrm{trace}(\Sigma)$. Therefore,

$$|||X|||_* = \mathrm{trace}(\Sigma) = \frac{1}{2}\left(|||C|||_F^2 + |||D|||_F^2\right).$$

$$\geq \min_{X=AB^T} \frac{1}{2}\left(|||A|||_F^2 + |||B|||_F^2\right).$$

$$\geq \min_{X=AB^T} |||A|||_F \cdot |||B|||_F.$$

- Next, we show $|||X|||_* \leq \min_{X=AB^T} |||A|||_F \cdot |||B|||_F$. We'll show it in two ways.

① For any A, B such that $X = AB^T$

$$|||X|||_* = |||AB^T|||_* \leq \langle \sigma(A), \sigma(B) \rangle \quad \text{(Horn \& Johnson, Thm 3.3.14(a))}$$

$$\leq \|\sigma(A)\|_2 \|\sigma(B)\|_2 = |||A|||_F \cdot |||B|||_F \quad \text{(Cauchy-Schwartz).}$$

Hence, $|||X|||_* \leq \min_{X=AB^T} |||A|||_F \cdot |||B|||_F$.

② For any A, B such that $X = AB^T = U\Sigma V^T$, then $U^TAB^TV = \Sigma$. Let ith row of U^TA and V^TB be c_i and d_i, respectively. Then. $\sigma_i = c_i d_i^T \leq \|c_i\|_2 \cdot \|d_i\|_2$

$$|||X|||_* = \mathrm{trace}(\Sigma) = \sum_{i=1}^r \sigma_i = \sum_{i=1}^r c_i d_i^T \leq \sum_{i=1}^r \|c_i\|_2 \cdot \|d_i\|_2$$

$$\leq \left(\sum_{i=1}^r \|c_i\|_2^2\right)^{\frac{1}{2}} \cdot \left(\sum_{i=1}^r \|d_i\|_2^2\right)^{\frac{1}{2}} = |||U^TA|||_F \cdot |||V^TB|||_F = \|A\|_F \cdot \|B\|_F$$