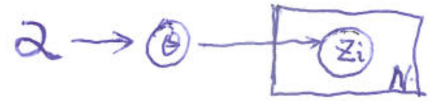
- Two ways to derive predictive distr. of multinomial-Dirichlet

Let $\theta \sim Dir(\alpha)$ be a Dirichlet r.v. in $\mathbb{R}^K$.

- $Z_1, \cdots, Z_N \sim \text{Multinomial}(\theta)$ be i.i.d. draw from multinomial with parameter θ.



Now let $Z_{new} \sim \text{Mutinomial}(\theta)$, we're interested in.

$$P(Z_{new} \mid Z_1, \cdots Z_N, \alpha).$$

① The posterior of θ given $Z_1, \cdots, Z_N$ is.

$$\theta \mid Z_1, \cdots Z_N \sim Dir(\alpha + \vec{y}) \quad \text{where } \vec{y} \in \mathbb{R}^K \text{ and } y_K = \sum_{i=1}^{N} \mathbb{1}(Z_i = K).$$

$$P(Z_{new} = j \mid Z_1, \cdots Z_N, \alpha) = \int P(Z_{new} = j \mid \theta) P(\theta \mid Z_1, \cdots Z_N, \alpha) d\theta$$

$$= \int \theta_j \cdot \frac{\Gamma(K\alpha + N)}{\prod_{K=1}^{K} \Gamma(\alpha + y_K)} \cdot \prod_{K=1}^{K} \theta_K^{y_K + \alpha_K - 1} \, d\theta.$$

$$= \frac{\Gamma(K\alpha + N)}{\prod_{K=1}^{K} \Gamma(\alpha + y_K)} \cdot \frac{\prod_{K \neq j} \Gamma(\alpha + y_K) \cdot \Gamma(\alpha + y_j)}{\Gamma(K\alpha + N + 1)} = \frac{y_j + \alpha}{K\alpha + N}.$$

② $$P(Z_1, \cdots, Z_N \mid \alpha) = \int P(Z_1, \cdots, Z_N \mid \theta) P(\theta \mid \alpha) d\theta$$

$$= \int \prod_{K=1}^{K} \theta_K^{y_K} \cdot \frac{1}{Z(\alpha)} \prod_{K=1}^{K} \theta_K^{\alpha_K - 1} d\theta.$$

$$= \frac{Z(\alpha + \vec{y})}{Z(\alpha)}$$

$$P(Z_{new} = j \mid Z_1, \cdots, Z_N, \alpha) = \frac{P(Z_{new} = j, Z_1, \cdots, Z_N \mid \alpha)}{P(Z_1, \cdots, Z_N \mid \alpha)} = \frac{Z(\alpha + \vec{y} + e_j)}{Z(\alpha + \vec{y})} = \frac{y_j + \alpha}{K\alpha + N}$$