Max-Cut

- Given a weighted undirected graph with n vertices, Want to find a cut $(S, \bar{S})$ so that the sum of weights across the cut is maximized.

- Define the indicator variable x_i, $i = 1, \cdots, n$.

$$x_i = \begin{cases} +1 & \text{if } i \in S \\ -1 & \text{if } i \in \bar{S} \end{cases}$$

- observe

$$x_i x_j = \begin{cases} +1 & \text{if } (i,j) \text{ is not in the cut} \\ -1 & \text{if } (i,j) \text{ is in the cut} \end{cases}$$

- Therefore

$$\frac{1}{2}(1 - x_i x_j) = \begin{cases} 0 & \text{if } (i,j) \text{ is not in the cut} \\ 1 & \text{if } (i,j) \text{ is in the cut} \end{cases}$$

- The Max-Cut problem can be written as

$$\begin{aligned} \max \quad & \frac{1}{4} \sum_{i,j} w_{ij} (1 - x_i x_j) \\ \text{s.t.} \quad & x_i \in \{-1, +1\}, \quad i = 1, \cdots, n \end{aligned}$$

- It's equivalent to

$$\begin{aligned} \max_{X, x} \quad & \frac{1}{4} \sum_{i,j} w_{ij} - \frac{1}{4} \operatorname{tr}(WX) \\ \text{s.t.} \quad & x \in \{-1, +1\}^n \\ & X = x x^T \end{aligned}$$

- Notice that

$$X = xx^T,\ x \in \{-1,+1\}^n \iff X_{ii} = 1,\ X \succeq 0,\ \mathrm{rank}(X)=1$$

proof: $\Rightarrow$: trivial.

$\Leftarrow$: $\begin{cases} \mathrm{rank}(X) = 1 \\ X \succeq 0 \end{cases}$ implies $X = xx^T$ for some x.

$X_{ii} = 1$ implies that $x_i^2 = 1$, i.e., $x_i \in \{+1,-1\}$.

- Hence, the Max-Cut can be written as

$$OPT = \max_{X \in S_n} \frac{1}{4}\sum_{i,j} w_{ij} - \frac{1}{4}\mathrm{tr}(WX).$$

$$\text{s.t. } X \succeq 0,\ X_{ii} = 1,\ i = 1, \cdots, n.$$
$$\mathrm{rank}(X) = 1.$$

- The SDP relaxation simply drops out the rank constraint.

$$SDP = \max_{X \in S_n} \frac{1}{4}\sum_{i,j} w_{ij} - \frac{1}{4}\mathrm{tr}(WX)$$

$$\text{s.t. } X \succeq 0$$
$$X_{ii} = 1,\ i = 1, \cdots, n.$$

- We have $SDP \geq OPT$. because SDP relaxation is maximizing over a larger constraint set.

- Observe that if the solution X^* to the SDP relaxation has rank 1, then SDP = OPT and X^* is also an optimal solution to OPT.
  In this case, $X^* = xx^T$ with $x \in \{-1, +1\}^n$, and x_i's give the optimal partition / cut.
- What if X^* has rank k with $k > 1$?
  Since $X^* \succeq 0$, $X^* = V^T V$ by the Cholesky decomposition with $V \in \mathbb{R}^{k \times n}$.

$$X^* = \begin{bmatrix} - & v_1^T & - \\ - & v_2^T & - \\ & \vdots & \\ - & v_n^T & - \end{bmatrix} \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}, \quad X^*_{ij} = v_i^T v_j.$$

Since $X^*_{ii} = 1$, $\|v_i\| = 1$, i.e., all v_i have the unit ℓ_2 norm. (If $k=1$, it is equivalent to $v_i = \pm 1$). It is the hope that in the case of $k > 1$, the nodes on the same side of cut have similar (or closer) v_i's.

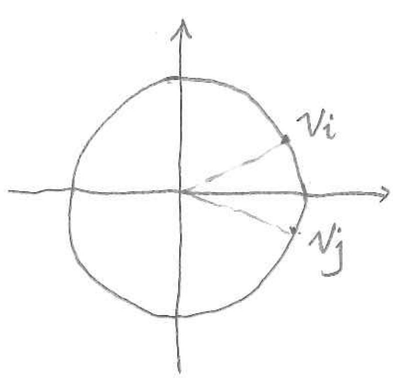

illustration for $k=2$.

- In the case of rank$(X^*) > 1$, Goemans and Williason (1995) came up the following SDP rounding scheme: choose a unit vector $v \in \mathbb{R}^k$ uniformly at random, and build the cut

$$S = \{ i \mid v^T v_i \geq 0 \} \quad \text{and} \quad \bar{S} = \{ i \mid v^T v_i < 0 \}$$

- By elementary orgoement,

$$\mathbb{P}\left(\operatorname{sign}(v^T v_i) \neq \operatorname{sign}(v^T v_j) \right) = \frac{\operatorname{acos}(v_i^T v_j)}{\pi}.$$

see Fig 4.3 in Ben-Tal & Nemirovski.

- Hence, the expected weight of the random cut is

$$\begin{aligned}
\mathbb{E}[\text{cut}] &\geq \sum_{i,j} \frac{W_{ij}}{2} \cdot \frac{\operatorname{acos}(v_i^T v_j)}{\pi} \\
&= \sum_{ij} \frac{W_{ij}}{2} \cdot \frac{\operatorname{acos}(X^*_{ij})}{\pi} \\
&= \sum_{ij} \frac{W_{ij}}{2} \cdot (1 - X^*_{ij}) \cdot \frac{\operatorname{acos}(X^*_{ij})}{\pi(1 - X^*_{ij})} \\
&\geq \sum_{ij} \frac{W_{ij}}{2} (1 - X^*_{ij}) \cdot \frac{\alpha}{2} \qquad (\alpha = 0.87856) \\
&= \frac{\alpha}{4} \sum_{ij} W_{ij} (1 - X^*_{ij}) \\
&= \alpha \cdot \text{SDP} \qquad (\text{page } 2).
\end{aligned}$$

- Since $\mathbb{E}[\text{cut}] \geq \alpha \cdot \text{SDP}$ and the cut for OPT can only be larger than this expectation, we have

$$\text{OPT} \geq \mathbb{E}[\text{cut}] \geq \alpha \cdot \text{SDP}.$$

- Comibining with $\text{SDP} \geq \text{OPT}$, we have

$$\text{SDP} \geq \text{OPT} \geq \mathbb{E}[\text{cut}] \geq \alpha \cdot \text{SDP} \geq \alpha \cdot \text{OPT}$$

- It implies two things.

① $$1 \geq \frac{\text{OPT}}{\text{SDP}} \geq \alpha = 0.87856.$$

② by using the randomized rounding scheme, we construct a random cut that, on average, has weight at least 0.87856 of OPT.