Matrix inversion lemma:

Consider the partition matrix $M = \begin{bmatrix} A & U \\ V & W \end{bmatrix}$ where A and W are assumed to invertible.

Now we consider two ways to "block-diagonalize" M.

① $\begin{bmatrix} I & 0 \\ -VA^{-1} & I \end{bmatrix} \cdot \begin{bmatrix} A & U \\ V & W \end{bmatrix} \cdot \begin{bmatrix} I & -A^{-1}U \\ 0 & I \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & W-VA^{-1}U \end{bmatrix}$

This corresponds to eliminate V before U.

② $\begin{bmatrix} I & -UW^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} A & U \\ V & W \end{bmatrix} \begin{bmatrix} I & 0 \\ -W^{-1}V & I \end{bmatrix} = \begin{bmatrix} A-UW^{-1}V & 0 \\ 0 & W \end{bmatrix}$

This correponds to eliminate U befre V.

Inverting ① gives us.

$$\begin{bmatrix} A & U \\ V & W \end{bmatrix}^{-1} = \begin{bmatrix} I & -A^{-1}U \\ 0 & I \end{bmatrix} \begin{bmatrix} A^{-1} & 0 \\ 0 & (W-VA^{-1}U)^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ -VA^{-1} & I \end{bmatrix}$$

$$= \begin{bmatrix} A^{-1} + A^{-1}U(W-VA^{-1}U)VA^{-1} & -A^{-1}U(W-VA^{-1}U)^{-1} \\ -(W-VA^{-1}U)VA^{-1} & (W-VA^{-1}U)^{-1} \end{bmatrix}$$

Inverting 3 gives us

$$\begin{bmatrix} A & U \\ V & W \end{bmatrix}^{-1} = \begin{bmatrix} I & 0 \\ -W^{-1}V & I \end{bmatrix} \begin{bmatrix} (A-UW^{-1}V)^{-1} & 0 \\ 0 & W^{-1} \end{bmatrix} \begin{bmatrix} I & -UW^{-1} \\ 0 & I \end{bmatrix}$$

$$= \begin{bmatrix} (A-UW^{-1}V)^{-1} & -(A-UW^{-1}V)^{-1}UW^{-1} \\ -W^{-1}V(A-UW^{-1}V)^{-1} & W^{-1}+W^{-1}V(A-UW^{-1}V)^{-1}UW^{-1} \end{bmatrix}$$

Therefore, setting the first equal gives us.

$$(A-UW^{-1}V)^{-1} = A^{-1}+A^{-1}U(W-VA^{-1}U)VA^{-1}$$

which is known as "Matrix inversion lemma".