① DP: given $(\mathcal{X}, \mathcal{F})$ and a finite measure α, DP is the distr. of a random prob measure P on $(\mathcal{X}, \mathcal{F})$ s.t. for any measurable partition $(A_1, \cdots, A_K)$ of $\mathcal{X}$.

$$(P(A_1), \cdots, P(A_K)) \sim Dir(\alpha(A_1), \cdots, \alpha(A_K)).$$

Thm: $P \sim DP(\alpha)$, $X_1, X_2, \cdots, X_n \overset{iid}{\sim} P$, then

$$P \mid X_1, \cdots, X_n \sim DP\left(\alpha + \sum_{i=1}^{n} \delta_{X_i}\right)$$

$\boxed{\text{posterior}}$

Prop: $(P_1, \cdots, P_K) \sim Dir(\alpha_1, \cdots, \alpha_K)$, and $X \sim Mul(P_1, \cdots, P_K)$, then

$$P_1, \cdots, P_K \mid X = j \sim Dir(\alpha_1, \cdots, \alpha_j + 1, \cdots, \alpha_K) = Dir(\alpha + e_j)$$

Prop: Joint distr of $(P_1, \cdots, P_K)$ and X is

$$\mathbb{P}(X = j, P_1 \le y_1, \cdots, P_K \le y_K)$$

$$= \int_0^{y_1} \cdots \int_0^{y_K} P_j \, d\, Dir(P_1, \cdots, P_K \mid \alpha) \quad ①$$

$$= \mathbb{P}(X = j)\, \mathbb{P}(P_1 \le y_1, \cdots, P_K \le y_K \mid X = j)$$

① = ②

$$= \mathbb{E}_P[\mathbb{P}(X = j \mid P)] \cdot Dir(y_1, \cdots, y_K \mid \alpha + e_j)$$

$$= \mathbb{E}_P[P_j] \; Dir(y_1, \cdots, y_K \mid \alpha + e_j)$$

$$= \frac{\alpha_j}{\alpha_\cdot} Dir(y_1, \cdots, y_K \mid \alpha + e_j) \quad ②$$

Prop: $P \sim DP(\alpha)$, $X \sim P$. Let $(B_1, \cdots, B_K)$ be a measurable partition of $\mathcal{X}$, and $A \in \mathcal{F}$, then

$$\mathbb{P}(X \in A, P(B_1) \le y_1, \cdots, P(B_K) \le y_K)$$

$$= \sum_{j=1}^{K} \frac{\alpha(A \cap B_j)}{\alpha(\mathcal{X})} Dir\Big(y_1, \cdots, y_K \,\Big|\, (\alpha(B_1), \cdots, \alpha(B_K)) + e_j\Big) \quad (*)$$

Proof: Define $\quad B_{j,1} = B_j \cap A, \quad B_{j,0} = B_j \cap A^c$

$$Y_{j,1} = P(B_j \cap A), \quad Y_{j,0} = P(B_j \cap A^c) \qquad P(B_j) = Y_{j,1} + Y_{j,0}.$$

$$\alpha_{j,1} = \alpha(B_j \cap A), \quad \alpha_{j,0} = \alpha(B_j \cap A^c).$$

$$\mathbb{P}(X \in A \mid Y_{j,0}, Y_{j,1}, j = 1, \dots, k) = \sum_{j=1}^{k} P(B_{j,1}) \doteq \sum_{j=1}^{k} Y_{j,1}$$

$$\mathbb{P}(X \in A, Y_{j,0} \le y_{j,0}, Y_{j,1} \le y_{j,1}, j = 1, \dots, k)$$

$$= \sum_{j=1}^{k} \int \cdots \int Y_{j,1} \, d\, Dir(Y_{1,0}, Y_{1,1}, \dots, Y_{k,0}, Y_{k,1} \mid \underbrace{\alpha_{1,0}, \alpha_{1,1}, \dots, \alpha_{k,0}, \alpha_{k,1}}_{\vec{\alpha}})$$

$$= \sum_{j=1}^{k} \frac{\overset{= \alpha(A \cap B_j)}{\alpha_{j,1}}}{\alpha(\mathcal{X})} Dir(y_{1,0}, y_{1,1}, \dots, y_{k,0}, y_{k,1} \mid \vec{\alpha} + e_{j,1})$$

By aggregation property of Dir distr. and $P(B_j) = Y_{j,0} + Y_{j,1}$, (*) holds. $\square$

Proof: (thm) WTS

$$P(B_1), \dots, P(B_k) \mid X \sim Dir(\alpha(B_1) + \delta_X(B_1), \dots, \alpha(B_k) + \delta_X(B_k)).$$

It's sufficient to check

$$\mathbb{P}(X \in A, P(B_1) \le y_1, \dots, P(B_k) \le y_k) = \int_A Dir(y_1, \dots, y_k \mid \alpha(B_1) + \delta_x(B_1), \dots, \alpha(B_k) + \delta_x(B_k)) \cdot \frac{d\alpha(x)}{\alpha(\mathcal{X})}$$

$$\text{RHS} = \sum_{j=1}^{k} \int_{A \cap B_j} Dir(y_1, \dots, y_k \mid (\alpha(B_1), \dots, \alpha(B_k)) + \delta_j) \cdot \frac{d\alpha(x)}{\alpha(\mathcal{X})}$$

$$= \sum_{j=1}^{k} \frac{\alpha(A \cap B_j)}{\alpha(\mathcal{X})} Dir(y_1, \dots, y_k \mid (\alpha(B_1), \dots, \alpha(B_k)) + e_j)$$

$$= \text{LHS} \quad (\text{by previous prop})$$

$\square$

Prop: $P \sim DP(\alpha)$, $X_1, \cdots, X_n \sim P$, then

$$\mathbb{P}(X_{n+1} \in A \mid X_1, \cdots, X_n) = \int P(A)\, d\,Dir(P(A), P(A^c) \mid X_1, \cdots, X_n)$$

$$= \frac{\alpha(A) + \sum_{i=1}^{n} \delta_{X_i}(A)}{\alpha(\mathcal{X}) + n}$$

$$= \frac{\alpha(\mathcal{X})}{\alpha(\mathcal{X})+n} \cdot \frac{\alpha(A)}{\alpha(\mathcal{X})} + \frac{n}{\alpha(\mathcal{X})+n} \cdot \frac{\sum_{i=1}^{n} \delta_{X_i}(A)}{n}$$

□

Define $\bar{\alpha}(A) = \frac{\alpha(A)}{\alpha(\mathcal{X})}$ be the normalized prob measure on $(\mathcal{X}, \mathcal{F})$.

Let $\theta_1, \theta_2, \cdots$, iid beta$(1, \alpha(\mathcal{X}))$ and $Y_1, Y_2, \cdots \overset{iid}{\sim} \bar{\alpha}$

define $\pi_1 = \theta_1$, $\pi_2 = \theta_2(1-\theta_1)$ $\cdots$, $\pi_K = \theta_n \prod_{i=1}^{K-1} (1-\theta_i)$, $\cdots$

and $P = \sum_{K=1}^{\infty} \pi_K \delta_{Y_K}$

Goal: $P \sim DP(\alpha)$.

$$P = \pi_1 \delta_{Y_1} + \sum_{K=2}^{\infty} \pi_K \delta_{\theta_K}$$

$$= \pi \delta_{Y_1} + (1-\theta_1) \sum_{K=2}^{\infty} \tilde{\pi}_K \delta_{Y_K}$$

$\tilde{\pi}_2 = \theta_2$
$\tilde{\pi}_3 = \theta_2(1-\theta_3)$
$\vdots$

$$P \overset{d}{=} \pi \delta_{Y_1} + (1-\theta_1) \cdot P \quad (*)$$

Idea: ① solution to $(*)$ is unique.

② finite Dirichlet measures satisfy $(*)$

① Lemma 1:
- W takes values in $[-1, 1]$
- U — — — in a linear space.
- V — — — same — — — as U
- $V \perp\!\!\!\perp (W, U)$
- $V \overset{d}{=} U + WV$
- $P(|W| = 1) \neq 1$.

$\Rightarrow$ uniqueness of V.

Proof Suppose V and V' are two different solution to
$V \stackrel{d}{=} U + WV$.

let (W_n, U_n) be indep. copies of (W, U).
and indep. of V, V'.

$V_1 = V, \quad V_1' = V'$

$V_{n+1} = U_n + W_n V_n, \quad V_{n+1}' = U_n + W_n V_n'$

then $V_{n+1} \stackrel{d}{=} V$ and $V_{n+1}' \stackrel{d}{=} V'$.

but $$|V_{n+1} - V_{n+1}'| = |W_n| \, |V_n - V_n'| = \prod_{i=1}^{n} |W_i| \cdot |V_1 - V_1'| \to 0.$$

contradiction that V and V' are different solution.

□

Lemma 2: $U \sim Dir(\alpha)$, $V \sim Dir(\gamma)$, $W \sim Beta(\alpha_\cdot, \gamma_\cdot)$

then $WU + (1-W)V \sim Dir(\alpha + \gamma)$

Lemma 3: $Dir(\alpha) \stackrel{d}{=} \sum_{j=1}^{k} \frac{\alpha_j}{\alpha_\cdot} Dir(\alpha + e_j)$

proof: let $U \sim Dir(\alpha)$, $X | U \sim Mul(U)$, $U | X = j \sim Dir(\alpha + e_j)$.

then $$\mathbb{P}(U \in C) = \mathbb{E}[\underbrace{\mathbb{P}(U \in C | X)}_{Dir(\alpha + e_X)(C)}]$$

$$= \sum_{j=1}^{k} \mathbb{P}(X = j) \, Dir(\alpha + e_j)(C)$$

$$= \sum_{j=1}^{k} \frac{\alpha_j}{\alpha_\cdot} Dir(\alpha + e_j)(C)$$

□

2. Proof: Pick a partition $(B_1, \cdots, B_k)$.

$V = (P(B_1), \cdots, P(B_k))$, $U = (\delta_{Y_1}(B_1), \cdots, \delta_{Y_1}(B_k))$

(*) becomes $V \stackrel{d}{=} \pi_1 U + (1 - \pi_1) V$

let $V \sim Dir(\alpha)$, where $\alpha = (\alpha(B_1), \cdots, \alpha(B_k))$.

Conditioning on the event $\{U = e_j\}$, i.e. $Y_1 \in B_j$.

$$\pi_1 U + (1-\pi_1) V \mid U = e_j \sim \pi_1 Dir(e_j) + (1-\pi_1) Dir(\alpha).$$

$$\sim Dir(\alpha + e_j).$$

Averaging over all possible event $\{U = e_j\}$

$$\mathbb{P}(U = e_j) = \mathbb{P}(Y_1 \in B_j) = \bar{\alpha}(B_j) = \frac{\alpha(B_j)}{\alpha(\mathcal{X})}.$$

The marginal distr. of $\pi_1 U + (1-\pi_1) V$ is.

$$\sum_{j=1}^{k} \frac{\alpha(B_j)}{\alpha(\mathcal{X})} Dir(\alpha + e_j) \stackrel{d}{=} Dir(\alpha).$$

This confirms that $Dir(\alpha)$ satisfy finite version of $(*)$.

$\square$.