Convergence of maximum of i.i.d r.v.s with finite second moment

Suppose $\{X_i\}_{i\geq 1}$ be a sequence of i.i.d r.vs with $\mathbb{E}[X_1^2]<\infty$.

Then $\dfrac{\max_{i=1,\dots,n} X_i}{\sqrt{n}} \xrightarrow{P} 0$, that is, $\max_{i=1,\dots,n} X_i = o_p(\sqrt{n})$.

Proof: $\mathbb{P}\left(\left|\dfrac{\max_i X_i}{\sqrt{n}}\right| \geq \varepsilon\right) \leq \mathbb{P}\left(\max_i |X_i| \geq \sqrt{n}\,\varepsilon\right)$

$$= \mathbb{P}\left(\max_i |X_i|^2 \geq n\varepsilon^2\right) \leq n\,\mathbb{P}\left(X_1^2 \geq n\varepsilon^2\right) \quad (*)$$

Now define $Y_n = n\,\mathbb{I}(X_1^2 \geq n\varepsilon^2)$, then we have.

① $Y_n \leq \dfrac{X_1^2}{\varepsilon^2}$ for all n. and $\dfrac{\mathbb{E}[X_1^2]}{\varepsilon^2} < \infty$

② $Y_n \xrightarrow{a.s.} 0$ because for any $w \in \Omega$, $\lim_{n\to\infty} n\,\mathbb{I}(X_1(w)^2 \geq n\varepsilon) = 0$

By DCT, $\mathbb{E}[Y_n] \to 0$. as $n \to \infty$. Therefore, in $(*)$

$$n\,\mathbb{P}(X_1^2 \geq n\varepsilon^2) = \mathbb{E}[Y_n] \to 0 \text{. as } n \to \infty$$

Alternative proof:

$$\mathbb{P}\left(\left|\frac{\max_i X_i}{\sqrt{n}}\right| \geq \varepsilon\right) \leq \mathbb{P}\left(\max_i |X_i| \geq \sqrt{n}\,\varepsilon\right).$$

$$= 1 - \mathbb{P}\left(\max_i |X_i| < \sqrt{n}\,\varepsilon\right) = 1 - \mathbb{P}(|X_1| < \sqrt{n}\,\varepsilon)^n.$$

$$= 1 - \mathbb{P}(X_1^2 < n\varepsilon^2)^n$$

From previous page, we know. $\mathbb{P}(X_1^2 \geq n\varepsilon^2) = o(\frac{1}{n})$, so

$$1 - \mathbb{P}(X_1^2 < n\varepsilon^2)^n = 1 - \left(1 - o(\tfrac{1}{n})\right)^n$$

$$= 1 - e^{-o(1)} \longrightarrow 1$$

$$\longrightarrow 1 - 1 = 0 \quad \text{as} \quad n \to \infty$$

Therefore, $\frac{\max_i X_i}{\sqrt{n}} \xrightarrow{P} 0.$